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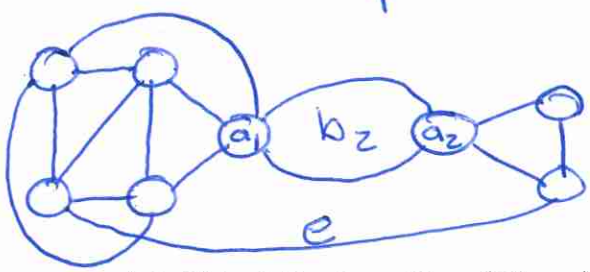
Graph Theory Quiz 5 (21 June 2019)
Open book, open notes, open neighbor.

1. For each of the following, prove that G must be planar or give a counter-example.

   (a) G is defined by $G = (G' + e)$, where G' can be defined by the block cut-point graph with blocks $\{b_1, b_2, b_3\}$ and articulation vertices $\{a_1, a_2\}$. a_1 connects blocks b_1 and b_2, a_2 connects blocks b_2 and b_3, and the edge e connects some vertex in b_1 to some vertex in b_3. b_1, b_2, and b_3 are all planar.

   Counter-example: e completes a Kuratowski subgraph

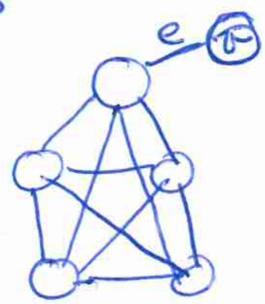


   (b) G is defined as $G = (G' - e)$, where G' is a minimal non-planar graph and $e = (u, v)$ is some edge $e \in E(G')$ such that $d(u) = d(v) = \Delta(G)$.

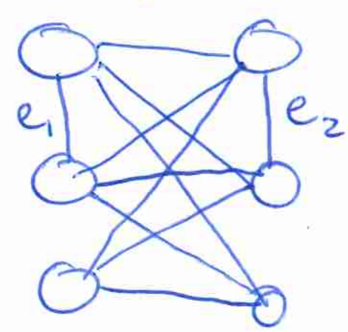
   G will always be planar by the definition of "minimal non-planar" for G'

   (c) G has $|V(G)| = 6$ and $|E(G)| = 11$.

   $K_5 + v + e$



   $K_{3,3} + e_1 + e_2$



2. Consider simple but possibly disconnected planar graph G where $|V(G)| \geq 2$. Prove that $\exists u, v \in V(G) : d(u) < 5, d(v) < 5$.

Consider the degree sum formula

$$\sum_{i}^{V(G)} d_i = 2m$$

and a necessary condition for planarity

$$m \leq 3n - 6$$

$$\sum_{i} d_i \leq 6n - 12$$

first assume in our degree sequence all vertices are of degree 6

$$\sum_{i} d_i = \underline{6n \leq 6n - 12} \rightarrow \text{obviously can't hold}$$

How many vertices of degree 5 or lesser do we need to hit the bound?

$$\sum_{i} d_i = 6(n - x) + 5x = 6n - 12$$

$$6n - x = 6n - 12$$

$$x = 12 \rightarrow \text{at least } 12$$

(if all other $d_i = 6$)